

SOLUTION EXERCISE SHEET 12

Exercise 1. We begin by recalling the definition of local uniform convergence.

Definition 0.1 (Local uniform convergence). Let $V \subseteq \mathbb{C}$ be an open set, $g_n : V \rightarrow \mathbb{C}$ be a sequence of functions and $g : V \rightarrow \mathbb{C}$ be a function. Then we say that g_n converges to g locally uniformly if for every $z \in V$ there exists $\epsilon > 0$ such that $D_\epsilon(z) \subseteq V$ and g_n converges uniformly on $D_\epsilon(z)$ to g .

Remark 0.2. Local uniform convergence as defined above is equivalent to uniform convergence of g_n to g on every compact subset $K \subseteq V$ (why?).

Now, observe that as $f_n : U \subseteq \mathbb{C} \rightarrow \mathbb{C}$ is a sequence of holomorphic functions, they are in particular continuous and as they converge locally uniformly to $f : U \subseteq \mathbb{C} \rightarrow \mathbb{C}$ it is clear that f is continuous (why?). To show that f is even holomorphic, we use the Morera Theorem, which tells us that a continuous function $g : V \subseteq \mathbb{C} \rightarrow \mathbb{C}$ on an open set $V \subseteq \mathbb{C}$ is holomorphic if for every contractible closed path $\gamma : [0, 1] \rightarrow V$ we have that $\int_\gamma g = 0$. But this is clearly satisfied for f as for any contractible path $\gamma : [0, 1] \rightarrow U$ and $n \in \mathbb{N}$ we have $\int_\gamma f_n = 0$ as the f_n are holomorphic and we have

$$\int_\gamma f_n \xrightarrow{n \rightarrow \infty} \int_\gamma f$$

as $\gamma([0, 1]) \subseteq U$ is a compact set and f_n converges uniformly to f on any compact subset of U (why does that imply the convergence of the integral?). We conclude that $f : U \rightarrow \mathbb{C}$ is holomorphic.

Lastly, we want to show that for all $k \in \mathbb{N}$, the sequence of the k -th derivatives converges locally uniformly to $f^{(k)}$. For that we fix $k \in \mathbb{N}$ and $z_* \in U$ and consider $r > 0$ such that $\overline{D_{2r}(z_*)} \subseteq U$. Then we know by Cauchy's integral formula that for any $z \in D_r(z_*)$ and any $n \in \mathbb{N}$ we have that

$$f_n^{(k)}(z) = \frac{k!}{2\pi i} \int_{\partial D_{2r}(z_*)} \frac{f_n(\zeta)}{(\zeta - z)^{k+1}} d\zeta \quad \text{and} \quad f^{(k)}(z) = \frac{k!}{2\pi i} \int_{\partial D_{2r}(z_*)} \frac{f(\zeta)}{(\zeta - z)^{k+1}} d\zeta.$$

Therefore we have that

$$|f_n^{(k)}(z) - f^{(k)}(z)| \leq 2r \cdot k! \sup_{\zeta \in \partial D_{2r}(z_*)} \left| \frac{f_n(\zeta) - f(\zeta)}{(\zeta - z)^{k+1}} \right| \leq \frac{k!}{r^k} \sup_{\zeta \in \partial D_{2r}(z_*)} |f_n(\zeta) - f(\zeta)|,$$

which tells us that

$$\sup_{z \in D_r(z_*)} |f_n^{(k)}(z) - f^{(k)}(z)| \leq \frac{k!}{r^k} \sup_{\zeta \in \partial D_{2r}(z_*)} |f_n(\zeta) - f(\zeta)|$$

and shows that $f_n^{(k)}$ converges locally uniformly to $f^{(k)}$ as f_n converges locally uniformly to f .

Remark 0.3. Observe that we only need $U \subseteq \mathbb{C}$ open. We do not need the information that U is connected.

Exercise 2. We begin by proving the first claim in the special case, i.e. under the additional hypothesis that $\lim_{\text{Im } z \rightarrow \pm\infty} f = 0$ we have that $|f| \leq 1$.

If $f = 0$, then clearly $|f| \leq 1$, else there exists $z_* \in U$ such that $|f(z_*)| > 0$. Let $C := |f(z_*)|$, then as $\lim_{\text{Im } z \rightarrow \pm\infty} f = 0$ we now that there exists $M > 0$ such that for all $z \in \bar{U}$ with $|\text{Im } z| \geq M$ we have that $|f(z)| < C$. Further, as f is continuous on \bar{U} , we have that it reaches its maximum on $[0, 1] \times [-Mi, Mi]$. Take $w \in [0, 1] \times [-Mi, Mi]$ a maximizer of $|f|$ and observe that by choice of $C, M > 0$ we have that $z_* \in [0, 1] \times (-Mi, Mi)$ giving that $|f(w)| \geq |f(z_*)| = C$ thus $w \in [0, 1] \times (-Mi, Mi)$. Further, w is clearly a maximizer of f on \bar{U} . Now, if $w \in (0, 1) \times (-Mi, Mi)$, then f has a local maximum in U and is therefore constant on U and by continuity on \bar{U} and we conclude that $f = C$ with $|C| \leq 1$. If $w \in \{0, 1\} \times (-Mi, Mi)$, then $|f(w)| \leq 1$, giving that $|f| \leq 1$. This proves the special case.

Let's consider now f as in the description of the exercise and suppose for contradiction that there exists $z_* \in U$ such that $|f(z_*)| > 1$. Then we define for any $\epsilon > 0$ the map $g_\epsilon(z) := \exp(\epsilon(z^2 - 1))f(z)$ and observe that $z^2 - 1 = x^2 - 1 - y^2 + 2ixy$ if we write $z = x + iy$. Therefore for any $z \in \bar{U}$ we have that $|g_\epsilon(z)| \leq \exp(-y^2)|f(z)| \leq |f(z)|$. From that we infer that $|g_\epsilon| \leq 1$ on the boundary of U and as f is bounded we have that $\lim_{\text{Im } z \rightarrow \pm\infty} g_\epsilon = 0$. We conclude that for any $\epsilon > 0$ we have $|g_\epsilon| \leq 1$. Now, as $|f(z_*)| > 1$ and

$$g_\epsilon(z_*) = \exp(\epsilon(z_*^2 - 1))f(z_*) \xrightarrow{\epsilon \rightarrow 0} f(z_*),$$

we can take $\epsilon > 0$ small enough such that $|g_\epsilon(z_*)| > 1$, which gives a contradiction. Thus we have that $|f| \leq 1$ on \bar{U} .

Finally, we want to show that we even have $|f| < 1$ on U if f is not constant. Suppose that f is not constant and that there exists $z_* \in U$ such that $|f(z_*)| = 1$. Then as we already now that $|f| \leq 1$ we have that f reaches its maximum in z_* . But then considering $[0, 1] \times [-Mi, Mi]$ with $M > |z|$ we get by the maximum principal and continuity that f is constant on $[0, 1] \times [-Mi, Mi]$. As we can take $M > 0$ arbitrarily big we conclude that f is constant. This is a contradiction, thus we have that $|f| < 1$ on U .

Exercise 3. (*) In this exercise we show that we need some kind of control on the growth of $|f|$ when $\text{Im } z \rightarrow \pm\infty$ in order to generalize the maximum principal to non bounded domains as we did in exercise 2.

In order to construct a function $f : \bar{U} \rightarrow \mathbb{C}$, which is continuous on \bar{U} , holomorphic on U and bounded by 1 on ∂U we first make the following observation. The

exponential is a map which is holomorphic on \mathbb{C} , bounded on $i\mathbb{R}$ and unbounded on each line $\{\lambda z \in \mathbb{C} \mid \lambda \in \mathbb{R}, z \in S^1 \setminus \{\pm i\}\}$. Considering only the half plane $\{\operatorname{Re}(z) > 0\}$ we have that \exp is unbounded but is bounded on $\{\operatorname{Re}(z) = 1\}$ by 1. The idea is use this information somehow on U , if we have a holomorphic map h on \bar{U} , which is such that $h(U) = \{\operatorname{Re}(z) > 0\}$ and $h(\partial U) \subseteq i\mathbb{R}$, we are done by composition with the exponential.

Next we observe that every $z \in \{\operatorname{Re}(z) \geq 0\} \setminus \{0\}$ can be written as $\exp(\lambda) \exp(\theta i) = \exp(\lambda + \theta i)$ for some $\lambda \in \mathbb{R}$ and $\theta \in [-\pi/2, \pi/2]$ and the domain $\mathbb{R} \times [-\pi/2, \pi/2]$ looks a lot like U .

Considering the function $z \mapsto \pi i z - \frac{\pi i}{2} = (x\pi - \frac{\pi}{2})i - y$ we have clearly a holomorphic map, which maps \bar{U} onto $\mathbb{R} \times [-\pi/2, \pi/2]$ and thus $z \mapsto \exp(\pi i z - \frac{\pi i}{2})$ maps \bar{U} onto $\{\operatorname{Re}(z) \geq 0\} \setminus \{0\}$.

Finally, we observe that $f(z) := \exp(-i \exp(\pi z i)) = \exp(-i \exp(-\pi y) \exp(\pi x i))$ is an entire map, which is such that on ∂U it is bounded by 1 and it is unbounded on U .

Exercise 4. Let p, q be two polynomials and assume that q is not the zero map. If q is a constant the claim is trivially respected. Suppose therefore that $\deg(q) \geq 1$. Further, let $\alpha_1, \dots, \alpha_l \in \mathbb{C}$ be the roots of q , i.e. all complex values such that $q(z) = 0$. Then, as q has only finitely many zeros, we have that around each $\alpha \in \{\alpha_1, \dots, \alpha_l\}$ there exists $\delta_\alpha > 0$ such that $q \neq 0$ on $D_{\delta_\alpha}(\alpha)^* := D_{\delta_\alpha}(\alpha) \setminus \{\alpha\}$. Further, as q is a polynomial, we can write $q(z) = \sum_{n=r_\alpha}^{\deg(q)} a_n (z - \alpha)^n$ with $a_{r_\alpha} \neq 0$ for some $r_\alpha \in \mathbb{N}_{\geq 1}$ as $q(\alpha) = 0$. Therefore, we can write $\frac{p(z)}{q(z)} = \frac{1}{(z-\alpha)^{r_\alpha}} \cdot h_\alpha(z)$ with a holomorphic map on $D_{\delta_\alpha}(\alpha)$. As h_α is holomorphic on $D_{\delta_\alpha}(\alpha)$ there exists a sequence (b_n) of complex values such that $h_\alpha(z) = \sum_{n=0}^{\infty} b_n (z - \alpha)^n$ on $D_{\delta_\alpha}(\alpha)$. We conclude that $\frac{p(z)}{q(z)} - \sum_{n=0}^{r_\alpha-1} b_n (z - \alpha)^{n-r_\alpha} = \sum_{n=0}^{\infty} b_{n+r_\alpha} (z - \alpha)^n$ has a holomorphic extension on $D_{\delta_\alpha}(\alpha)$.

More generally, doing the above around each α_j we get $r_1, \dots, r_l \in \mathbb{N}$ and β_{kj} with $j = 1, \dots, l$ and $k(j) = 1, \dots, r_j$ such that

$$\frac{p(z)}{q(z)} - \sum_{j=1}^l \sum_{k=1}^{r_j} \frac{\beta_{kj}}{(z - \alpha_j)^k}$$

has a holomorphic extension on \mathbb{C} (why?). Let's call this holomorphic extension h and observe that h equals the above expression on $\mathbb{C} \setminus \{\alpha_1, \dots, \alpha_l\}$. In particular, h equals the above expression if $|z| \geq M$ for $M > 0$ large enough. But as p is a polynomial there exists clearly $m \in \mathbb{N}$ such that $\frac{h(z)}{z^m} \xrightarrow{|z| \rightarrow \infty} 0$. Thus $|h(z)| \leq C|z|^m$ for some $C > 0$ and all $|z| \geq M$ (why?). Then by the same argument as in exercise 2 sheet 11 we conclude that h is a polynomial of degree at most m . We infer that

$$\frac{p(z)}{q(z)} = h(z) + \sum_{j=1}^l \sum_{k=1}^{r_j} \frac{\beta_{kj}}{(z - \alpha_j)^k}$$

on $\mathbb{C} \setminus \{\alpha_1, \dots, \alpha_l\}$ with h a polynomial. This proves the claim.

Remark 0.4. Observe that if $\deg(p) = \deg(q)$ then h is a non zero constant and if $\deg(q) > \deg(p)$ we have that $h = 0$.